THE MALGRANGE-EHRENPREIS THEOREM IN DISTRIBUTION THEORY

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ABSTRACT

ERIC CHIN HOOI FU: The Malgrange-Ehrenpreis Theorem in Distribution Theory
(Under the direction of Mark Williams.)

This thesis explores the Malgrange-Ehrenpreis Theorem in the theory of distribution via an expository approach based on the text “Introduction to the Theory of Distribution” by G. Friedlander and M. Joshi. I will be focusing on the understanding of the proof of the Malgrange-Ehrenpries Theorem. In many places, I have closely followed the presentation of Friedlander's and Joshi's monograph.

We begin with the definition and the properties of distributions. Next, we study Fourier transforms and its properties. We look at the Schwartz space and consider its dual space. We conclude the section by investigating the Fourier transform of several notable distributions, including the Dirac-delta distribution and the signum distribution.

Finally, we introduce Fourier-Laplace transforms and examine the Paley-Wiener-Schwartz estimate. This estimate describes the polynomial decay property of the Fourier-Laplace transform of test functions. We conclude with a special case of Malgrange-Ehrenpreis Theorem and analyze its complex proof.

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Introduction

Distributions are objects which are more general than functions. For this, they are sometimes called *generalized functions*. We shall discuss how distributions are accounted discontinuous solutions in a simple ordinary differential equation.

Consider the following ordinary differential equation on $\mathbb{R}$:

$$y'(x) - ay(x) = 0, \quad a \in \mathbb{R}. \tag{1}$$

Let $y \in C^1(\mathbb{R})$ and φ be infinitely differentiable on $\mathbb{R}$ with compact support. Then

$$\int_{\mathbb{R}} (y'(x) - ay(x))\, \varphi(x)\ \mathrm{d}x = \int_{\mathbb{R}} y'(x)\varphi(x)\ \mathrm{d}x - a \int_{\mathbb{R}} y(x)\varphi(x)\ \mathrm{d}x = 0, \tag{2}$$

whence, via integration by parts, the first integral reduces to

$$\int_{\mathbb{R}} y'(x)\varphi(x)\ \mathrm{d}x = \ y(x)\varphi(x)\Big|_{\mathbb{R}} - \int_{\mathbb{R}} y(x)\varphi'(x)\ \mathrm{d}x = -\int_{\mathbb{R}} y(x)\varphi'(x)\ \mathrm{d}x. \tag{3}$$

We remark that the evaluation of the term $y\varphi$ is zero because φ has compact support in $\mathbb{R}$. Hence

$$-\int_{\mathbb{R}} y(x)\varphi'(x)\ \mathrm{d}x - a \int_{\mathbb{R}} y(x)\varphi(x)\ \mathrm{d}x = 0. \tag{4}$$

Equivalently,

$$\int_{\mathbb{R}} y(x)\, (\varphi'(x) + a\varphi(x))\ \ \mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}). \tag{5}$$

If $y \in L^1(\mathbb{R})$ is discontinuous and Equation (5) holds for all $\varphi \in C_c^\infty(\mathbb{R})$, then we say that y is a distribution solution to (1).

Definition and Properties of Distribution

Several Notations and Definitions

Let $X \subset \mathbb{R}^n$. The support of a function $f : X \to \mathbb{C}$ is the closure of the set on which $f(x) \neq 0$. We write supp f for this. Suppose $f \in C^\infty(X)$. We write the derivatives as

$$\partial_j f = \frac{\partial f}{\partial x_j}, \quad j = 1, 2, \ldots, n.$$

For higher order derivatives, we shall employ the multi-index notation. Let the n-tuplet $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index; we say that its order is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then we have

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

so that $\partial^\alpha \partial^\beta f = \partial^{\alpha+\beta} f$. Finally, with the convention $\alpha! = \alpha_1! \cdots \alpha_n!$, we have Taylor's Theorem as

$$f(x+h) = \sum_{\alpha \geq 0} \frac{x^\alpha}{\alpha!} \partial^\alpha f(h),$$

where we set $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

Test Functions and Distributions

Let $X \subset \mathbb{R}^n$. Test functions are functions that behave "sufficiently good." They have continuous derivatives of all order with compact support (that is, these functions vanish outside of some bounded regions). We denote the vector space of test functions as $C_c^\infty(X)$. For example,

$$\varphi(x) = \begin{cases} \exp\left[\dfrac{1}{(x-a)(x-b)}\right], & a < x < b; \\[2ex] 0, & \text{elsewhere.} \end{cases} \tag{6}$$

is a member of $C_c^\infty(\mathbb{R}^n)$ with supp $\varphi = (a, b)$. Next, we introduce the notion of convergence in the space of test functions as a definition:

Definition 1. *Let $X \subset \mathbb{R}^n$ be an open set. A sequence of test functions $\{\varphi_j\}_{j=1}^{\infty}$ is said to converge to zero in $C_c^\infty(X)$ if*

1. $\operatorname{supp} \varphi_j \subset K$*, whereby K is a fixed compact subset of X,*
2. *for each multi-index α, the derivatives $\partial^\alpha \varphi_j$ converges to zero uniformly as $j \to \infty$.*

We note that this is a stringent definition, which is much stronger than ordinary convergence. Now, we are now ready to define distributions.

Definition 2. *Let $X \subset \mathbb{R}^n$ be an open set. A linear form $u : C_c^\infty(X) \to \mathbb{C}$ is called a **distribution** if, for every compact set $K \subset X$, there exist a positive real number C and a nonnegative integer N such that*

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi| \tag{7}$$

for all $\varphi \in C_c^\infty(X)$ with $\operatorname{supp} \varphi \subset K$. This inequality is called a semi-norm estimate.

The vector space of distributions on X is denoted as $\mathcal{D}'(X)$. The notation $\langle u, \varphi \rangle$ indicates the "action" of the distribution u on the test function φ. Occasionally this action is referred to as a *distributional pairing* of $u : \varphi \mapsto \langle u, \varphi \rangle$. The object $\langle u, \varphi \rangle$ is a member of $\mathbb{C}$.

Every locally integrable function f generates a distribution f via the following theorem.

Theorem 3. *Let $X \subset \mathbb{R}^n$ be an open set and let $f \in C^0(X)$. Then*

$$\langle f, \varphi \rangle = \int_X f(x)\varphi(x) \, \mathrm{d}x, \quad \varphi \in C_c^\infty(X) \tag{8}$$

is a distribution. We sometimes speak of the distribution determined by f.

Proof. This follows from the following estimate

$$|\langle f, \varphi \rangle| = \left| \int_X f\varphi \, \mathrm{d}x \right| \leq \int_X |f||\varphi| \, \mathrm{d}x \leq \sup_{x \in K} |\varphi| \int_K |f| \, \mathrm{d}x, \quad \varphi \in C_c^\infty(K), \tag{9}$$

where $K \subset X$ is a compact set. □

We remark that the theorem above gives an injective map $C^0(X) \to \mathcal{D}(X)$. Another common example of distribution is the Dirac-delta distribution δ given by

$$\langle \delta, \varphi \rangle = \varphi(0), \quad \varphi \in C_c^\infty(\mathbb{R}^n). \tag{10}$$

More generally, let $X \subset \mathbb{R}^n$. A "translated" Dirac-delta distribution is a member of $\mathcal{D}'(X)$ defined by $\langle \delta_y, \varphi \rangle = \varphi(y)$, where $y \in X$ and φ is a test function in X. We shall record a property of the Dirac-delta distribution. To ease exposition, denote the reflection map (about the origin) with the symbol $\check{}$, that is, $\check{f}(x) = f(-x)$.

Proposition 4. *Let δ be the Dirac-delta distribution. Then $\check{\delta} = \delta$.*

Proof. Let φ be any test function. Then

$$\left\langle \check{\delta}, \varphi \right\rangle = \left\langle \delta, \check{\varphi} \right\rangle = \check{\varphi}(0) = \varphi(0) = \left\langle \delta, \varphi \right\rangle. \qquad \square$$

In the next theorem, we shall characterize distributions differently by appealing to the notion of convergence. This property is called *sequential continuity.*

Theorem 5. *A linear form $u : C_c^\infty(X) \to \mathbb{C}$ is a distribution if and only if for every sequence of test functions $\{\varphi_j\}$ which converges to zero in $C_c^\infty(X)$ as $j \to \infty$, we have*

$$\lim_{j\to\infty} \left\langle u, \varphi_j \right\rangle = 0. \tag{11}$$

Proof. Suppose u is a distribution. Then we have the following semi-norm estimate

$$0 \leq | \left\langle u, \varphi_j \right\rangle | \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_j|.$$

Passing the limit as $j \to \infty$, we get $\lim \left\langle u, \varphi \right\rangle = 0$.

To prove the converse direction, we shall proceed by contradiction. Assume that u is sequentially continuous but not a member of $\mathcal{D}'(X)$. Then the semi-norm estimate in Equation (7) does not hold. In particular, there exists a compact set $K \subset X$ such that, for each nonnegative integer N, the set of numbers

$$\left\{ \frac{| \left\langle u, \varphi \right\rangle |}{\displaystyle\sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi|} \right\}, \quad \varphi \in C_c^\infty(K) \tag{12}$$

are unbounded. It follows that for each $N = 0, 1, \ldots$, there exists $\varphi_N \in C_c^\infty(K)$ such that

$$\left\langle u, \varphi_N \right\rangle \geq N \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_N|. \tag{13}$$

Now we define

$$\Phi_N = \frac{\varphi_N}{N \displaystyle\sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_N|}. \tag{14}$$

Then $\operatorname{supp}\Phi_N \subset K$ and $|\partial^\beta \Phi_N| \leq 1/N$ for $|\beta| \leq N$. By definition, this sequence of test functions $\{\Phi_N\}$ tends to zero in $C_c^\infty(X)$ as $N \to \infty$. However, by Equation (13), we see that

$$|\langle u, \Phi_N \rangle| = \left| \left\langle u, \frac{\varphi_N}{N \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_N|} \right\rangle \right| = \frac{\langle u, \varphi_N \rangle}{N \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_N|} \geq \frac{N \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_N|}{N \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_N|} = 1 \quad (15)$$

for all N. This is a contradiction, hence completing the proof of the converse. □

We shall now briefly discuss the notion of convergence of distributions.

Definition 6. *Let $X \subset \mathbb{R}^n$ be an open set and let $\{u_j\}_{j=1}^\infty$ be a sequence of distributions on X. The sequence converges in $\mathcal{D}'(X)$ to $u \in D'(X)$ if*

$$\lim_{j \to \infty} \langle u_j, \varphi \rangle = \langle u, \varphi \rangle \quad \textit{for all } \varphi \in C_c^\infty(X). \quad (16)$$

We remark that the notion of distributional convergence should be distinguished from that of the usual convergence. Consider $f \in C^0(\mathbb{R})$ with $\operatorname{supp} f \subset [0,1]$ such that $\int f(x)\ \mathrm{d}x = 1$. Define $f_k(x) = kf(kx)$ for $k = 1, 2, \ldots$. Fix $x \in \mathbb{R}$ and we see that $f_k(x) \to 0$ pointwise as $k \to \infty$. Indeed, assume that x is nonzero (otherwise it is trivial). For large k, we have $|kx| \gg 1$, thus falling outside of the compact support. But, if φ is a test function on $\mathbb{R}$,

$$|\langle f_k, \varphi \rangle - \varphi(0)| = \left| \int_{\mathbb{R}} f_k(x) \left(\varphi(x) - \varphi(0)\right)\ \mathrm{d}x \right| \leq k \int_{\mathbb{R}} |f(kx)||\varphi(x) - \varphi(0)|\ \mathrm{d}x. \quad (17)$$

Now, via a change of variable $\xi = kx$, we see that $|\langle f_k, \varphi \rangle - \varphi(0)|$ is bounded from above by

$$\int_0^1 |f(\xi)| \left| \varphi\left(\frac{\xi}{k}\right) - \varphi(0) \right|\ \mathrm{d}\xi = \left(\int |f|\ \mathrm{d}x \right) \left(\sup \left\{ |\varphi(x) - \varphi(0)| : 0 \leq x \leq \frac{1}{k} \right\} \right). \quad (18)$$

Therefore f_k converges to the Dirac-delta distribution δ in the distributional sense as $k \to \infty$.

We shall end this section with a brief discussion on odd distributions. In the sense of classical functions, we say that f is odd if and only if $\check{f} = -f$. It follows that

$$\int_{\mathbb{R}} f(x)\varphi(-x)\ \mathrm{d}x = \int_{\mathbb{R}} f(-x)\varphi(x)\ \mathrm{d}x = -\int_{\mathbb{R}} f(x)\varphi(x)\ \mathrm{d}x \quad (19)$$

if $f(x)$ is an odd function. This motivates the following definition:

Definition 7. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then u is an odd distribution if and only if*

$$\langle u, \check{\varphi} \rangle = -\langle u, \varphi \rangle \quad \textit{for all } \varphi \in C_c^\infty(\mathbb{R}^n). \quad (20)$$

Operations on Distributions

In this section, we study multiplication on distributions as well as distributional derivatives. Let $X \subset \mathbb{R}^n$ be an open set. We define the product of a distribution and a smooth function as follows:

Product of a distribution and a smooth function

Definition 8. *If $u \in \mathcal{D}'(X)$ and $f \in C^\infty(X)$, then, for $\varphi \in C_c^\infty(X)$,*

$$\langle fu, \varphi \rangle = \langle u, f\varphi \rangle \tag{21}$$

is a member of $\mathcal{D}'(X)$ and is called the product of f and u.

This definition is consistent with the case when u is a continuous function. Formally,

$$\langle fu, \varphi \rangle = \int fu\varphi \, \mathrm{d}x = \langle u, f\varphi \rangle .$$

Note that the second member of Equation (21) is well-defined. This is because if φ is a test function and f is smooth, the product $f\varphi$ is also a test function. Now it remains to show that fu is a distribution. By virtue of Leibniz's Theorem, if $K \subset X$ is a compact set and $\varphi \in C_c^\infty(K)$, then there exist constants $C_0, C_1, \ldots$, depending on f and K but not φ, such that, for $N = 0, 1, \ldots$

$$\sum_{|\alpha| \leq N} \sup |\partial^\alpha (f\varphi)| \leq C_N \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi|, \quad \varphi \in C_c^\infty(K), \tag{22}$$

which is the semi-norm estimate for fu that we seek. We conclude that multiplication $u \mapsto fu$ is a map $\mathcal{D}'(X) \to \mathcal{D}'(X)$.

This definition allows us to study an interesting corollary:

Corollary 9. *Suppose δ is the Dirac-delta distribution in $\mathbb{R}^n$. Then, for any $x \in \mathbb{R}^n$,*

$$x\delta(x) = 0. \tag{23}$$

Proof. Since $x \mapsto x$ is a smooth function, we have

$$\langle x\delta(x), \varphi(x) \rangle = \langle \delta(x), x\varphi(x) \rangle = \left. x\varphi(x) \right|_{x=0} = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$ □

The derivatives of a distribution

To motivate the notion of derivatives of a distribution, suppose $u \in C^1(X)$ and consider the distribution determined by the derivatives $\partial_i u$, with $i = 1, \ldots, n$. Performing integration by parts yields

$$\langle \partial_i u, \varphi \rangle = \int \varphi \partial_i u \; \mathrm{d}x = -\int u \partial_i \varphi \; \mathrm{d}x = -\langle u, \partial_i \varphi \rangle, \quad i = 1, \ldots, n, \quad \varphi \in C_c^\infty(X). \tag{24}$$

The last member in the equation above makes sense for $u \in \mathcal{D}'(X)$ since the derivatives of test functions are also test functions. In fact, we have linearity, supp $\partial^\alpha \varphi \subset$ supp φ for all multi-indicies α, and the semi-norm estimate: if $K \subset X$ is compact,

$$|\langle u, \partial_i \varphi \rangle| \leq C \sum_{|\alpha| \leq N+1} \sup |\partial^\alpha \varphi|, \quad \varphi \in C_c^\infty(K). \tag{25}$$

These imply that if $u \in \mathcal{D}'(X)$, then so is $\partial_i u$. In other words, distributional differentiation is a map $\mathcal{D}'(X) \to \mathcal{D}'(X)$. We shall formalize this in the following definition:

Definition 10. *Let $u \in \mathcal{D}'(X)$. The derivative map $\partial^\alpha : \mathcal{D}'(X) \to \mathcal{D}'(X)$, where α is a multi-index, is defined by*

$$\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle, \quad \varphi \in C_c^\infty(X). \tag{26}$$

As we have seen earlier, this definition is consistent with the case when u is a continuous and differentiable function. Next, we shall examine an example involving the Heaviside function in $X = \mathbb{R}$:

$$H(x) = \begin{cases} 1, & x > 0; \\ 0, & x \leq 0. \end{cases} \tag{27}$$

Applying Definition 10, we have, for all $\varphi \in C_c^\infty(\mathbb{R})$,

$$\langle \partial H, \varphi \rangle = -\langle H, \partial \varphi \rangle = -\int_0^\infty \partial \varphi(x) \; \mathrm{d}x = \varphi(0),$$

whence we conclude that

$$\partial H = \delta. \tag{28}$$

The next example is the principal value distribution $1/x$ defined as

$$\left\langle \frac{1}{x}, \varphi(x) \right\rangle = \lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{-\epsilon} \frac{\varphi(x)}{x} \; \mathrm{d}x + \int_{\epsilon}^{\infty} \frac{\varphi(x)}{x} \; \mathrm{d}x \right), \quad \varphi \in C_c^\infty(\mathbb{R}). \tag{29}$$

We shall show that the limit exists and that it is a distribution. Integration by parts yields

$$\int \frac{\varphi(x)}{x}\,\mathrm{d}x = \varphi(x)\ln|x| - \int \varphi'(x)\ln|x|\,\mathrm{d}x.$$

Therefore

$$\lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty}\frac{\varphi(x)}{x}\,\mathrm{d}x = -\varphi(\epsilon)\ln|\epsilon| - \int_{\epsilon}^{\infty}\varphi'(x)\ln|x|\,\mathrm{d}x, \tag{30}$$

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x = \varphi(-\epsilon)\ln|-\epsilon| - \int_{-\infty}^{-\epsilon}\varphi'(x)\ln|x|\,\mathrm{d}x. \tag{31}$$

We remark that the terms evaluated at infinity vanish because they fall outside of the compact support of the test function φ. Summing these two equations gives the principal value distribution:

$$\left\langle \frac{1}{x},\varphi(x)\right\rangle = (\ln\epsilon)\left[\varphi(-\epsilon)-\varphi(\epsilon)\right] - \left[\int_{-\infty}^{-\epsilon}\varphi'(x)\ln|x|\,\mathrm{d}x + \int_{\epsilon}^{\infty}\varphi'(x)\ln|x|\,\mathrm{d}x\right]. \tag{32}$$

By the Mean Value Theorem,

$$\varphi(-\epsilon)-\varphi(\epsilon) = \varphi'(\omega\epsilon)(-\epsilon-\epsilon) = -2\epsilon\varphi'(\omega\epsilon) \tag{33}$$

for some $-1<\omega<1$. But φ' is bounded, hence, by L'Hôpitals' Rule,

$$(\ln\epsilon)\left[\varphi(-\epsilon)-\varphi(\epsilon)\right] = -2\epsilon\varphi'(\omega\epsilon)\ln\epsilon \to 0 \text{ as } \epsilon\to 0. \tag{34}$$

Thus, as $\epsilon\to 0$,

$$\left\langle \frac{1}{x},\varphi(x)\right\rangle \to -\int_{\mathbb{R}}\varphi'(x)\ln|x|\,\mathrm{d}x, \quad \varphi\in C_c^{\infty}(\mathbb{R}). \tag{35}$$

We have shown that the limit in Equation (29) exists. It remains to show that it is a member of $\mathcal{D}'(\mathbb{R})$. Let $K\subset\mathbb{R}$ be a compact subset containing the support of $1/x$. Then

$$\left|\left\langle \frac{1}{x},\varphi(x)\right\rangle\right| \le \int_K |\varphi'(x)|\,|\ln|x||\,\mathrm{d}x \le \sup|\varphi'|\int_K |\ln|x||\,\mathrm{d}x \tag{36}$$

serves as the semi-norm estimate of $1/x$, rendering $1/x\in\mathcal{D}'(\mathbb{R})$.

We conclude this section with a proposition involving d'Alembert's equation.

Proposition 11. *Let $f\in C^0(\mathbb{R})$. Then f satisfies the d'Alembert's equation:*

$$(\partial_t+\partial_x)f(x-t) = 0. \tag{37}$$

Proof. For starters, consider $g \in C^1(\mathbb{R})$. Let $s = x - t$. Then by the chain rule, we have

$$(\partial_t + \partial_x)g(s) = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} = \frac{\partial g}{\partial s} \cdot \frac{\partial s}{\partial t} + \frac{\partial g}{\partial s} \cdot \frac{\partial s}{\partial x} = -\frac{\partial g}{\partial s} + \frac{\partial g}{\partial s} = 0. \tag{38}$$

Now suppose $\varphi \in C_c^\infty(\mathbb{R}^2)$. If $g \in C^1(\mathbb{R})$, it follows that, by Definition 10,

$$\langle(\partial_t + \partial_x)g, \varphi\rangle = -\langle g, (\partial_t + \partial_x)\varphi\rangle = -\iint_{\mathbb{R}^2} g(x-t)\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial t}\right)\, \mathrm{d}x\ \mathrm{d}t = 0. \tag{39}$$

But all these arguments hold for $g \in \mathbb{C}^1(\mathbb{R})$. To deal with $f \in C^0(\mathbb{R})$, we shall consider the following idea: we construct a sequence of functions $\{f_n\}$ in $C^1(\mathbb{R})$ such that $f_n \to f$ in $C^0(\mathbb{R})$. This can be achieved via the Weierstrass Polynomial Approximation Theorem, which asserts that for $f \in C([a,b])$, where $[a,b] \subset \mathbb{R}$ is a compact interval, there exists a sequence of polynomials $\{p_n\}$ such that

$$\sup_{x \in [a,b]} |p_n - f| \to 0 \text{ as } n \to \infty. \tag{40}$$

Pick $\{f_n\} = \{p_n\}$. It remains to establish that

$$f_n(s) \to f(s) \text{ as } n \to \infty. \tag{41}$$

Let $K \subset \mathbb{R}^2$ be a compact subset containing the point (x,t). The map $K \to \mathbb{R}$ via

$$\xi : (x,t) \to x - t \tag{42}$$

is continuous. Since K is compact, the image of K under the aforementioned map is, in particular, compact. Let $I \subset \mathbb{R}$ be a compact interval containing $\xi(x,t) = x - t$. Pick the interval $[a,b] \subset I$, as in the hypothesis of the Weierstrass Polynomial Approximation Theorem.

Now fix $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $n > N$ implies that

$$|p_n - f| \le \epsilon, \quad x \in [a,b]. \tag{43}$$

Or

$$|p_n| \le |f| + \epsilon \quad \text{for all } n > N. \tag{44}$$

We can choose $|f| + \epsilon$ to dominate the sequence of functions $\{p_n\}$, provided $n > N$. Pick

$$f_1 = p_N,\ f_2 = p_{N+1},\ f_3 = p_{N+2}, \ldots,\ f_k = p_{N+k-1}. \tag{45}$$

Finally, we invoke the Lebesgue Dominated Convergence Theorem; since for $\epsilon > 0$, we have

$|f_k| \leq |f| + \epsilon$, then

$$\begin{aligned}
\langle(\partial_t + \partial_x)f(x-t), \varphi(x,t)\rangle &= \left\langle(\partial_t + \partial_x)\lim_{n\to\infty} f_n(x-t), \varphi(x,t)\right\rangle \\
&= -\left\langle \lim_{n\to\infty} f_n(x-t), (\partial_t + \partial_x)\varphi(x,t)\right\rangle \\
&= -\iint_{\mathbb{R}^2} \lim_{n\to\infty} f_n(x,t)\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial t}\right)\,\mathrm{d}x\;\mathrm{d}t \\
&= -\lim_{n\to\infty}\iint_{\mathbb{R}^2} f_n(x,t)\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial t}\right)\,\mathrm{d}x\;\mathrm{d}t \\
&= -\lim_{n\to\infty}\langle f_n(x-t), (\partial_t + \partial_x)\varphi(x,t)\rangle \\
&= \lim_{n\to\infty}\langle(\partial_t + \partial_x)f_n(x-t), \varphi(x,t)\rangle \\
&= 0.
\end{aligned}$$

□

Test Functions with Parameter Dependence

We begin with the introduction of the vector space of continuous linear forms on $C^\infty(X)$, where $X \subset \mathbb{R}^n$. This space is denoted as $\mathcal{E}'(X)$.

Definition 12. *Let $X \subset \mathbb{R}^n$ be an open set. A linear form on u on $C^\infty(X)$ is called continuous if there are a compact set $K \subset X$, a nonnegative constant C, and a nonnegative integer N such that*

$$|\langle u, \varphi\rangle| \leq C \sum_{|\alpha|\leq N} \sup_{x\in K} |\partial^\alpha\varphi|, \quad \varphi \in C^\infty(X). \tag{46}$$

We shall conclude this chapter with the statements pertaining to test functions which depend on a parameter. The latter result is a special case of the former. These assertions are stated without proof:

Theorem 13. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open sets. Let u be an element of $\mathcal{D}'(X)$. Suppose that $\varphi \in C^\infty(X \times Y)$ with the following property: every point $y' \in Y$ has a neighborhood $U(y')$ in Y such that the support of $x \mapsto \varphi(x,y)$ is contained in a compact set $K = K(y')$ is $y \in U(y')$. Then*

$$\langle u(x), \varphi(x,y)\rangle \in C^\infty(Y) \tag{47}$$

and

$$\partial^\alpha \langle u(x), \varphi(x,y)\rangle = \left\langle u(x), \partial_y^\alpha \varphi(x,y)\right\rangle \tag{48}$$

for all multi-indices α.

Corollary 14. *Let $u \in \mathcal{E}'(X)$ and $\varphi \in C^\infty(X \times Y)$. Then $\psi = \langle u(x), \varphi(x,y)\rangle \in C^\infty(Y)$.*

Fourier Transforms

Properties of Fourier Transforms

The Fourier transform of a function f on $\mathbb{R}^n$ is

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int f(x)e^{-ix\cdot\xi}\ \mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{49}$$

whereby $x \cdot \xi$ is the usual Euclidean inner product

$$x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j \tag{50}$$

To ensure the existence of the integral, we may pick $f \in L^1(\mathbb{R}^n)$, that is f such that

$$||f||_{L^1} = \int |f(x)|\ \mathrm{d}x < \infty.$$

We shall study some elementary properties of Fourier transforms on $L^1(\mathbb{R}^n)$, which are collected together in the following theorem. Prior to doing so, we shall recall that the convolution of two functions f and g on $\mathbb{R}^n$ is the function

$$(f * g)(x) = \int f(y)g(x-y)\ \mathrm{d}y = \int f(x-y)g(y)\ \mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{51}$$

The existence of the integrals can be realized by assuming that both functions are continuous and that one of them is compactly supported.

Theorem 15. *Let $f \in L^1(\mathbb{R}^n)$. Then $\hat{f}$ exists and is continuous. Moreover, $\hat{f}$ is bounded by the L^1-norm of f. Furthermore, suppose that $g \in L^1(\mathbb{R}^n)$, then*

$$\int f(x)\hat{g}(x)\ \mathrm{d}x = \int \hat{f}(\xi)g(\xi)\ \mathrm{d}\xi \tag{52}$$

and $f * g$ is a member of $L^1(\mathbb{R}^n)$, with the following Fourier transform

$$\widehat{f * g} = \hat{f}\hat{g}. \tag{53}$$

Proof. The existence of $\mathcal{F}f = \hat{f}$ is due to the fact that f is an L^1 function on $\mathbb{R}^n$, as discussed above. The bound of $\hat{f}$ follows from

$$|\hat{f}(\xi)| = \left| \int f(x) e^{-ix\cdot\xi} \, \mathrm{d}x \right| \leq \int |f(x)||e^{-ix\cdot\xi}| \, \mathrm{d}x = \int |f(x)| \, \mathrm{d}x = ||f||_{L^1}, \quad \xi \in \mathbb{R}^n. \tag{54}$$

To establish continuity, let $\xi, h \in \mathbb{R}^n$ be arbitrary. Then

$$\left| \hat{f}(\xi + h) - \hat{f}(\xi) \right| = \left| \int_{-\infty}^{\infty} f(x) e^{-ix\cdot\xi} \left(e^{-ix\cdot h} - 1 \right) \, \mathrm{d}x \right| \leq \int_{-\infty}^{\infty} |f(x)| \, |e^{-ix\cdot h} - 1| \, \mathrm{d}x.$$

Note that since the integrand is dominated by an integrable function, that is

$$|f(x)| \, |e^{-ix\cdot h} - 1| \leq 2|f(x)|,$$

and that for any sequence $\{h_n\}$, the sequence $f_n(x) = f(x)(e^{-ix\cdot h_n} - 1) \to 0$ as $h_n \to 0$, then by the Lebesgue Dominated Convergence Theorem we are able to pass the following limit:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x)| \, |e^{-ix\cdot h_n} - 1| \, \mathrm{d}x = \int_{-\infty}^{\infty} \lim_{n\to\infty} |f(x)| \, |e^{-ix\cdot h_n} - 1| \, \mathrm{d}x = 0. \tag{55}$$

But $\{h_n\}$ is an arbitrary null sequence. It follows that

$$\lim_{h\to 0} \left| \hat{f}(\xi + h) - \hat{f}(\xi) \right| = 0, \tag{56}$$

that is, $\hat{f}$ is uniformly continuous. In particular, $\hat{f}$ is a continuous function of $\xi \in \mathbb{R}^n$.

To prove Equation (52), we apply Fubini's Theorem:

$$\begin{aligned}
\int \hat{f}(\xi) g(\xi) \, \mathrm{d}\xi &= \int \left(\int f(x) e^{-ix\cdot\xi} \, \mathrm{d}x \right) g(\xi) \, \mathrm{d}\xi \\
&= \int f(x) \left(\int g(\xi) e^{-i\xi\cdot x} \, \mathrm{d}\xi \right) \, \mathrm{d}x \\
&= \int f(x) \hat{g}(x) \, \mathrm{d}x
\end{aligned}$$

Since both φ and ψ decrease rapidly at infinity, the middle integral converges absolutely, justifying the change of order of integration.

The convolution $f * g$ is a member of $L^1(\mathbb{R}^n)$ because its L^1-norm

$$\int |f * g| \, \mathrm{d}x = \int \left| \int f(y)g(x-y) \, \mathrm{d}y \right| \, \mathrm{d}x \leq \int \int |f(y)g(x-y)| \, \mathrm{d}x \, \mathrm{d}y = ||f||_{L_1} ||g||_{L_1}. \tag{57}$$

is finite, for both f and g are members of $L^1(\mathbb{R}^n)$. Finally, Equation (53) can be established in the following manner:

$$\begin{aligned} \int \widehat{f * g} \, \mathrm{d}x &= \int e^{-ix\cdot\xi} \, \mathrm{d}x \int f(y)g(x-y) \, \mathrm{d}y \\ &= \int f(y) \, \mathrm{d}y \int g(x-y)e^{-ix\cdot\xi} \, \mathrm{d}x \\ &= \int f(y) \, \mathrm{d}y \int g(z)e^{-i(y+z)\cdot\xi} \, \mathrm{d}z \\ &= \int f(y)e^{-iy\cdot\xi} \, \mathrm{d}y \int g(z)e^{-iz\cdot\xi} \, \mathrm{d}z \\ &= \hat{f}\hat{g} \end{aligned}$$

We remark that there is a change of variable $x = y + z$ in the third equality. $\square$

Schwartz Space

We shall adopt the differential operators $\mathcal{D}_j = -i\partial_j$, $j = 1, \ldots, n$, hereafter. The new space of test functions which we are going to study is the *Schwartz space*, whose members are called rapidly decreasing test functions.

Definition 16. *A function $\varphi \in C^\infty(\mathbb{R}^n)$ is rapidly decreasing if the semi-norm*

$$||\varphi||_{\alpha,\beta} = \sup |x^\alpha D^\beta \varphi| < \infty \tag{58}$$

holds for all pairs of multi-indices α and β.

The vector space of such functions is called the Schwartz space, denoted $\mathcal{S}(\mathbb{R}^n)$. Rapidly decreasing test functions are sometimes called *Schwartz functions* for their membership. We remark that an element $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a smooth function such that φ and $D^\alpha\varphi$ decay faster than any polynomial at infinity.

We formulate the notion of convergence in the Schwartz space as follows:

Definition 17. *A sequence of rapidly decreasing test functions $\{\varphi_j\}_{j=1}^\infty$ is said to converge to zero in $\mathcal{S}(\mathbb{R}^n)$ if $||\varphi||_{\alpha,\beta} \to 0$ as $j \to \infty$ for all α and β.*

Several consequences of the definition will be stated without proof:

Proposition 18. *Let φ be a Schwartz function. Then*

1. *$\mathcal{S}(\mathbb{R}^n)$ is stable under differentiation and multiplication by polynomials. Moreover, if p and q are polynomials, then $\varphi \mapsto p(x)q(D)\varphi$ is a continuous map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.*

2. *$C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.*

3. *$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and both injections are continuous.*

These properties yield two important identities:

Lemma 19. *If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\widehat{D^\alpha \varphi} = \xi^\alpha \hat{\varphi}(\xi) \tag{59}$$

and

$$\widehat{x^\alpha \varphi} = (-1)^{|\alpha|} D^\alpha \hat{\varphi}(\xi). \tag{60}$$

Proof. Both Fourier transforms exist by the continuous injections of $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. To prove Equation (59) and Equation (60) respectively, we integrate by parts and differentiate under the integral sign. When

$$\widehat{D^\alpha \varphi}(\xi) = \int D^\alpha \varphi(x) e^{-ix\cdot\xi} \, \mathrm{d}x = \xi \int D^{\alpha-1} \varphi(x) e^{-ix\cdot\xi} \, \mathrm{d}x. \tag{61}$$

is iterated, we get

$$\widehat{D^\alpha \varphi}(\xi) = \xi^\alpha \int \varphi(x) e^{-ix\cdot\xi} \, \mathrm{d}x = \xi^\alpha \hat{\varphi}(\xi). \tag{62}$$

This proves Equation (59). To establish Equation (60), we shall investigate its second member:

$$D^\alpha \hat{\varphi}(\xi) = D^\alpha \int \varphi(x) e^{-ix\cdot\xi} \, \mathrm{d}x = (-1)^{|\alpha|} \int x^\alpha \varphi(x) e^{-ix\cdot\xi} \, \mathrm{d}x = (-1)^{|\alpha|} \widehat{x^\alpha \varphi}. \tag{63}$$

Here, $D^\alpha = D_\xi^\alpha$. Dividing both sides of Equation (63) by $(-1)^{|\alpha|}$ proves Equation (60). □

The following lemma allows us to bound the L^1-norm of a Schwartz function with the sum of its semi-norms.

Lemma 20. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$||\varphi||_{L^1(\mathbb{R}^n)} \leq C \sum_{|\alpha| \leq 2N} ||\varphi||_{\alpha,0}. \tag{64}$$

Proof. The continuous injection of $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$, as stated in Proposition 18, allows us to speak of the L^1-norm of φ. By definition,

$$||\varphi||_{L^1(\mathbb{R}^n)} = \int |\varphi| \; \mathrm{d}x = \int |\varphi| \, (1+|x|)^N \, (1+|x|)^{-N} \; \mathrm{d}x \tag{65}$$

Since φ is a Schwartz function, it follows that

$$\sup \left\{ |\varphi| \, (1+|x|)^N \right\} \tag{66}$$

is finite; moreover, by ensuring that $N > n$,

$$||\varphi||_{L^1(\mathbb{R}^n)} \leq \sup \left\{ |\varphi| \, (1+|x|)^N \right\} \int (1+|x|)^{-N} \; \mathrm{d}x \tag{67}$$

makes sense because the integral converges. Finally, since the integral is itself finite, we obtain the estimate

$$||\varphi||_{L^1(\mathbb{R}^n)} \leq C \sum_{|\alpha| \leq 2N} ||\varphi||_{\alpha,0}, \tag{68}$$

for some constant C, whereby

$$||\varphi||_{\alpha,0} = \sup |x^\alpha \varphi|$$

is the semi-norm for Schwartz functions. □

Theorem 21. *The Fourier transform is a continuous map $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.*

Proof. Consider the Fourier transform of $D^\alpha(x^\beta \varphi)$, where α and β are multi-indices. By Lemma 19, we have

$$\widehat{D^\alpha(x^\beta \varphi)} = \xi^\alpha \widehat{x^\beta \varphi} = (-1)^{|\beta|} \xi^\alpha D^\beta \hat{\varphi}. \tag{69}$$

We note that by Proposition 18, both $x^\beta \varphi$ and $D^\alpha(x^\beta \varphi)$ are members of the Schwartz space. Equation (69) allows us to write the semi-norm of $\hat{\varphi}$ as

$$||\hat{\varphi}||_{\alpha,\beta} = \sup |\xi^\alpha D^\beta \hat{\varphi}| = \sup |(-1)^{|\beta|} \widehat{D^\alpha(x^\beta \varphi)}| < \infty, \tag{70}$$

since $D^\alpha(x^\beta \varphi) \in \mathcal{S}(\mathbb{R}^n)$. This implies that $\hat{\varphi}$ satisfies Definition 16, rendering $\hat{\varphi}$ an element of the Schwartz space.

We have shown that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$; so it remains to establish the continuity of $\mathcal{F}$. Due to the continuous injection of $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$, Equation (54) allows us to continue working on

Equation (70) in the following manner:

$$||\hat{\varphi}||_{\alpha,\beta} = \sup |\widehat{D^\alpha(x^\beta\varphi)}| \leq ||D^\alpha(x^\beta\varphi)||_{L^1(\mathbb{R}^n)} \leq C \sum_{|\gamma|\leq 2N} ||D^\alpha(x^\beta\varphi)||_{\gamma,0}. \tag{71}$$

The last equality is due to Lemma 20. Lastly, we employ Leibniz's Theorem to get another upper bound:

$$||\hat{\varphi}||_{\alpha,\beta} \leq C' \sum_{\substack{|\delta|\leq|\beta|+2N \\ |\omega|\leq|\alpha|}} ||\varphi||_{\delta,\omega}. \tag{72}$$

Here, C' is a different constant. We conclude from the estimate above that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a continuous map. □

Corollary 22.

$$\mathcal{F}\left[\exp\left(-\frac{|x|^2}{2}\right)\right] = (2\pi)^{n/2} \exp\left(-\frac{|\xi|^2}{2}\right), \quad x \in \mathbb{R}^n. \tag{73}$$

Proof. We shall deduce this corollary from the identities in Equation (59) and Equation (60). When $n = 1$, then $\varphi(x) = \exp(-|x|^2/2)$ satisfies the differential equation

$$iD\varphi + x\varphi = 0.$$

By virtue of Equation (59) and Equation (60), we have

$$\mathcal{F}\left(iD\varphi + x\varphi\right) = i\xi\hat{\varphi}(\xi) - D\hat{\varphi}(\xi) = 0. \tag{74}$$

This in turn produces a differential equation for $\hat{\varphi}$:

$$iD\hat{\varphi} + \xi\hat{\varphi} = 0. \tag{75}$$

By separation of variables, we get

$$\int \frac{\mathrm{d}\hat{\varphi}}{\hat{\varphi}} = \int \xi \ \mathrm{d}\xi,$$

whence we obtain the solution $\hat{\varphi}(\xi) = A\exp(-\xi^2/2)$ for some constant of integration A. To determine the value of A, we appeal to the Gaussian integral

$$A = \hat{\varphi}(0) = \int e^{-x^2/2} \ \mathrm{d}x = \sqrt{2\pi}. \tag{76}$$

Therefore,

$$\hat{\varphi}(\xi) = \sqrt{2\pi}\exp(-\xi^2/2). \tag{77}$$

Finally, we see that Equation (73) holds because

$$\mathcal{F}\left[\exp\left(-\frac{|x|^2}{2}\right)\right] = \prod_{j=1}^{n} \int e^{-ix_j\xi_j - x_j^2/2} \, \mathrm{d}x_j = \prod_{j=1}^{n} \hat{\varphi}(\xi_j).$$

That is, for $x \in \mathbb{R}^n$,

$$\mathcal{F} : \exp\left(-\frac{|x|^2}{2}\right) \mapsto (2\pi)^{n/2} \exp\left(-\frac{|\xi|^2}{2}\right), \tag{78}$$

as desired. □

This is an important corollary because we have shown that, with appropriate normalization of the $(2\pi)^{n/2}$ factor, the Gaussian function

$$x \mapsto \exp\left(-\frac{|x|^2}{2}\right) \tag{79}$$

is a fixed point of the Fourier transform in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Before we proceed to the Fourier Inversion Formula, we shall introduce a lemma:

Lemma 23. *If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, then*

$$\widehat{\tau_{-h}\varphi} = \hat{\varphi}(\xi)e^{i\xi\cdot h}. \tag{80}$$

Proof. By definition in Equation (49), we have

$$\hat{\varphi}(\xi) = \int \varphi(x)e^{-ix\cdot\xi} \, \mathrm{d}x.$$

It follows that, via the change of variable of $y = x - h$,

$$\hat{\varphi}(\xi)e^{i\xi\cdot h} = \int \varphi(x)e^{-ix\cdot\xi}e^{i\xi\cdot h} \, \mathrm{d}x = \int \varphi(x)e^{-i\xi\cdot(x-h)} \, \mathrm{d}x = \int \varphi(y+h)e^{-iy\cdot\xi} \, \mathrm{d}y = \widehat{\tau_{-h}\varphi}. \quad \square$$

Theorem 24 (Fourier Inversion Formula)**.** *If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\varphi(x) = (2\pi)^{-n} \int \hat{\varphi}(\xi)e^{ix\cdot\xi} \, \mathrm{d}\xi. \tag{81}$$

Proof. Let φ and ψ be rapidly decreasing test functions. Then, by Equation (52), we have

$$\int \hat{\varphi}(\xi)\psi(\xi) \, \mathrm{d}\xi = \int \varphi(x)\hat{\psi}(x) \, \mathrm{d}x.$$

Let $\epsilon > 0$ and put $\psi(\xi) = \exp(-\epsilon^2|\xi|^2/2)$. By Corollary 22,

$$\int \hat{\varphi}(\xi) \exp(-\epsilon^2|\xi|^2/2) \, \mathrm{d}\xi = \frac{(2\pi)^{n/2}}{\epsilon^n} \int \varphi(x) \exp(-|x|^2/2\epsilon^2) \, \mathrm{d}x = (2\pi)^{n/2} \int \varphi(\epsilon z) \exp(-|z|^2/2) \, \mathrm{d}z.$$

Here, we let $x = \epsilon z$. Now, we shall apply Lebesgue Dominated Convergence Theorem on both integrals. For the integral on the left hand side, we remark that $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and $0 < \exp(-\epsilon^2|\xi|^2/2) \leq 1$; hence we may pass the limit $\epsilon \to 0$ through the integral sign. On the other hand, for the integral on the right hand side, we note that $\exp(-|z|^2/2) \in L^1(\mathbb{R}^n)$ and $\varphi(\epsilon z)$ is bounded; thus we may pass the limit $\epsilon \to 0$ through the integral sign. These yield

$$\int \hat{\varphi}(\xi) \, \mathrm{d}\xi = (2\pi)^{n/2} \varphi(0) \int \exp(-|x|^2/2) \, \mathrm{d}x,$$

from which

$$\varphi(0) = (2\pi)^{-n} \int \hat{\varphi}(\xi) \, \mathrm{d}\xi. \tag{82}$$

Now set $\psi(x) = \varphi(x + y)$ for $y \in \mathbb{R}^n$. Then, by Lemma 23, we have $\hat{\psi}(\xi) = \hat{\varphi}(\xi) e^{iy\cdot\xi}$, whence

$$\varphi(y) = \psi(0) = (2\pi)^{-n} \int \hat{\psi}(\xi) \, \mathrm{d}\xi = (2\pi)^{-n} \int \hat{\varphi}(\xi) e^{iy\cdot\xi} \, \mathrm{d}\xi. \tag{83}$$

Since $y \in \mathbb{R}^n$ is arbitrary, the proof of the Fourier Inversion Formula is complete. □

Theorem 25. *The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a continuous isomorphism, so is its inverse $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.*

Proof. We assert that $\mathcal{F}$ is injective, for if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{\varphi} = 0$, Equation (81) implies that $\varphi = 0$. Surjectivity follows from the following formulation of Equation (81):

$$\varphi(x) = (2\pi)^{-n} \int \hat{\varphi}(-\xi) e^{-ix\cdot\xi} \, \mathrm{d}\xi = \hat{\psi}(x), \tag{84}$$

whereby $\psi(\xi) = (2\pi)^{-n} \hat{\varphi}(-\xi)$, which is a rapidly decreasing test function in $\mathbb{R}^n$. Finally, the bijection, coupled with Theorem 21, we see that $\mathcal{F}$ is a continuous isomorphism. The continuity of $\mathcal{F}^{-1}$ also follows from Equation (84). □

In fact, we shall see that Equation (84) admits a more concise form:

$$\varphi(x) = \hat{\psi}(x) = (2\pi)^{-n} \widehat{\hat{\varphi}(-\xi)} = (2\pi)^{-n} \hat{\hat{\varphi}}(-x),$$

whence

$$\hat{\hat{\varphi}} = (2\pi)^n \check{\varphi}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{85}$$

Tempered Distributions

We wish to study the dual of Schwartz space $\mathcal{S}(\mathbb{R}^n)$, as we did with the space of test functions $C_c^\infty(\mathbb{R}^n)$. The dual of $\mathcal{S}(\mathbb{R}^n)$ is a set of continuous linear forms on $\mathcal{S}(\mathbb{R}^n)$; this space is denoted as $\mathcal{S}'(\mathbb{R}^n)$, and its members are called *tempered distributions.*

Definition 26. *A linear form u on $\mathcal{S}(\mathbb{R}^n)$ is a tempered distribution if and only if there exist a nonnegative constant C and a nonnegative integer N such that*

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha|,|\beta| \leq N} \sup |x^\alpha D^\beta \varphi|, \quad \varphi \in \mathcal{S}(R^n). \tag{86}$$

An example of a tempered distribution is a continuous function f of polynomial growth, i.e. a function that satisfies

$$|f(x)| \leq C(1+|x|)^M, \quad x \in \mathbb{R}^n \tag{87}$$

for some nonnegative constants C and M.

We proceed to the Fourier transform on the space of tempered distributions.

Definition 27. *The Fourier transform of $u \in \mathcal{S}'(\mathbb{R}^n)$ is the distribution $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ given by*

$$\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{88}$$

The second member of Equation (88) makes sense as we are pairing a tempered distribution u with a rapidly decreasing test function $\hat{\varphi}$; this is due to Theorem 21 — the Fourier transform is a map $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Moreover, this definition is consistent with the case when $u \in L^1(\mathbb{R}^n)$:

$$\langle \hat{u}, \varphi \rangle = \int \hat{u}(\xi)\varphi(\xi) \, \mathrm{d}\xi = \int u(x)\hat{\varphi}(x) \, \mathrm{d}x = \langle u, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The property used is discussed in Equation (52).

Theorem 28. *The Fourier transform is an isomorphism $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.*

Proof. By Definition 88 and Equation (85),

$$\left\langle \hat{\hat{u}}, \varphi \right\rangle = \langle \hat{u}, \hat{\varphi} \rangle = \left\langle u, \hat{\hat{\varphi}} \right\rangle = (2\pi)^n \langle u, \check{\varphi} \rangle,$$

whence we obtain

$$\langle u, \check{\varphi} \rangle = (2\pi)^{-n} \langle \hat{u}, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{89}$$

We assert that $\mathcal{F}$ is an injective map, for if $\hat{u} = 0$, we have $u = 0$. Moreover, $\mathcal{F}$ is surjective.

From Equation (89), we may write

$$\langle u, \varphi \rangle = \left\langle (2\pi)^{-n} (\hat{u})^{\check{}}, \hat{\varphi} \right\rangle, \tag{90}$$

By Equation (88), we conclude that u is the Fourier transform of $(2\pi)^{-n}(\hat{u})^{\check{}}$. $\square$

We shall now list some operational rules on the space of tempered distributions collectively as a theorem — they are not foreign as we have seen them in Lemma 19 under the Schwartz space setting.

Theorem 29. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then*

$$\widehat{D^\alpha u} = \xi^\alpha \hat{u} \tag{91}$$

and

$$\widehat{x^\alpha u} = (-1)^{|\alpha|} D^\alpha \hat{u}. \tag{92}$$

Proof. Suppose $u \in \mathcal{S}(\mathbb{R}^n)$. Then, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\langle \widehat{D^\alpha u}, \varphi \right\rangle = \langle D^\alpha u, \hat{\varphi} \rangle = \left\langle u, (-1)^{|\alpha|} D^\alpha \hat{\varphi} \right\rangle = \left\langle u, \widehat{x^\alpha \varphi} \right\rangle = \langle \hat{u}, x^\alpha \varphi \rangle = \langle \xi^\alpha \hat{u}, \varphi \rangle. \tag{93}$$

The first equality is from the definition of Fourier transform on the space of tempered distributions; the second equality is due to the definition of distributional derivatives; the third equality holds by virtue of Equation (60). Hence we have established that

$$\widehat{D^\alpha u} = \xi^\alpha \hat{u}.$$

Using a similar approach,

$$\left\langle \widehat{x^\alpha u}, \varphi \right\rangle = \langle x^\alpha u, \hat{\varphi} \rangle = \langle u, \xi^\alpha \hat{\varphi} \rangle = \left\langle u, \widehat{D^\alpha \varphi} \right\rangle = \langle \hat{u}, D^\alpha \varphi \rangle = \left\langle (-1)^{|\alpha|} \hat{u}, \varphi \right\rangle \tag{94}$$

for all Schwartz functions φ. Thus we have proven that

$$\widehat{x^\alpha u} = (-1)^{|\alpha|} D^\alpha \hat{u}. \qquad \square$$

A classical example in Fourier transform involves the Dirac-delta distribution on $\mathbb{R}^n$:

$$\left\langle \hat{\delta}, \varphi \right\rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int \varphi(x) \, \mathrm{d}x = \langle 1, \varphi \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{95}$$

whence

$$\hat{\delta} = 1. \tag{96}$$

Now, applying Fourier transform on both sides yields

$$\hat{1} = \hat{\hat{\delta}} = (2\pi)^n \check{\delta} = (2\pi)^n \delta, \tag{97}$$

where we have used the fact that $\check{\delta} = \delta$ (from Proposition 4) and Equation (85).

Next, let us consider the signum function on $\mathbb{R}$:

$$\text{sign } x = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases} \tag{98}$$

Recall that, from Equation (28), the distributional derivative of the Heaviside function H is the Dirac-delta function δ. Hence H is the distributional antiderivative of δ. We then write

$$\int_{-\infty}^{x} \delta(t) \text{ d}t = H(x) \equiv \frac{1 + \text{sign } x}{2}. \tag{99}$$

Here, the integral is taken in the distributional sense. It follows that

$$\delta = \partial H = \partial \left(\frac{1 + \text{sign } x}{2} \right),$$

whence $\partial(\text{sign } x) = 2\delta$. Applying Fourier transform on both sides, we get

$$i\xi \, \widehat{\text{sign } x} = \mathcal{F}\left[\partial(\text{sign } x)\right] = 2\hat{\delta} = 2.$$

Therefore,

$$\widehat{\text{sign } x} = \frac{2}{i\xi} + A\delta(\xi), \tag{100}$$

whereby A is a constant and $1/\xi$ is the principal value distribution, as seen in Equation (29). The term $A\delta(\xi)$ in Equation (100) is recorded to ensure that we have the most general expression for $\widehat{\text{sign } x}$. For, if we were to multiply Equation (100) by $i\xi$ on both sides, we get

$$i\xi \, \widehat{\text{sign } x} = 2 + 2i\xi\delta(\xi) = 2.$$

The second summand of the second member in the equation above vanishes due to Corollary 9.

Since the signum function is odd (in the sense of Definition 7), it follows that $A = 0$. We conclude that

$$\widehat{\text{sign } x} = \frac{2}{i\xi}. \tag{101}$$

Now, applying Equation (101) on Equation (99) yields

$$\widehat{H} = \frac{1}{2}\left(\hat{1} + \widehat{\text{sign } x}\right) = \frac{1}{2}\left(2\pi\delta + \frac{2}{i\xi}\right) = \pi\delta + \frac{1}{i\xi}. \tag{102}$$

From Equation (101), we have

$$\frac{1}{x} = \frac{i}{2}\,\widehat{\text{sign}}\,(x).$$

Applying Fourier transform on both sides yields

$$\widehat{1/x} = \frac{i}{2}\,\widehat{\widehat{\text{sign}}}\,(\xi) = \frac{i}{2}\cdot 2\pi \text{ sign } (-\xi),$$

where the second equality is due to Equation (85). Finally, since the signum function is odd,

$$\widehat{1/x} = -i\pi \text{ sign } \xi. \tag{103}$$

We summarize our findings as follow:

$$\begin{aligned}
\hat{\delta} &= 1 \\
\hat{1} &= (2\pi)^n\delta \\
\widehat{1/x} &= -i\pi \text{ sign } (\xi) \\
\widehat{\text{sign}}\,(x) &= \frac{2}{i\xi}
\end{aligned}$$

The Malgrange-Ehrenpreis Theorem

Fourier-Laplace Transforms

The Laplace transform of a function on the real line is

$$\int f(x)e^{-px}\,\mathrm{d}x, \quad p \in \mathbb{C}.$$

We note that setting $p = i\zeta$ reduces the Laplace transform to the Fourier transform:

$$\int f(x)e^{-i\zeta x}\,\mathrm{d}x. \tag{104}$$

Equation (104) is called the *Fourier-Laplace transform* of f when we restrict $\zeta \in \mathbb{R}$.

The Paley-Wiener-Schwartz Lemma

Definition 30. *If $u \in \mathcal{E}'(\mathbb{R}^n)$, then*

$$\hat{u}(\zeta) = \left\langle u(x), e^{-ix\cdot\zeta}\right\rangle, \quad \zeta \in \mathbb{C}^n \tag{105}$$

is the Fourier-transform of u.

When u is a test function, the equation in the definition above reduces to the more familiar integral form

$$\hat{u}(\zeta) = \int u(x)e^{-ix\cdot\zeta}\,\mathrm{d}x, \quad \zeta \in \mathbb{C}^n.$$

This definition bears some consequences which are the main ingredients of the Paley-Wiener-Schwartz Lemma.

Lemma 31 (Paley-Wiener-Schwartz Lemma)**.** *Several properties of $\hat{u}$ follow from the definition above:*

1. *The Fourier-Laplace transform of $u \in \mathcal{E}'(\mathbb{R}^n)$ is analytic on $\mathbb{C}^n$.*

2. *Let a be a positive real number. If $u \in \mathcal{E}'(\mathbb{R}^n)$ and* supp $u \subset \{|x| \leq a\}$*, then there exist nonnegative constants C and N such that*

$$|\hat{u}(\zeta)| \leq C\,(1+|\zeta|)^N\, e^{a|\mathrm{Im}\ \zeta|}, \quad \zeta \in \mathbb{C}^n, \tag{106}$$

where $|\zeta| = \sqrt{|\zeta_1|^2 + \cdots + |\zeta_n|^2}$.

3. *If $u \in C_c^\infty(\mathbb{R}^n)$ and* supp $u \subset \{|x| \leq a\}$*, then there exist nonnegative constants C_m, $m = 0, 1, \ldots$, such that*

$$|\hat{u}(\zeta)| \leq C_m\,(1+|\zeta|)^{-m}\, e^{a|\mathrm{Im}\ \zeta|}, \quad \zeta \in \mathbb{C}^n,\ m = 0, 1, \ldots \tag{107}$$

Proof. By Corollary 14, we have $\hat{u} \in C^\infty(\mathbb{C}^n)$. Moreover, $\zeta \mapsto e^{-ix\cdot\zeta}$ is analytic on $\mathbb{C}^n$. Thus $\hat{u}$ satisfies the Cauchy-Riemann equations.

Let $\psi \in C^\infty(\mathbb{R})$ be such that

$$\psi(t) = \begin{cases} 1, & t \geq -\frac{1}{2}; \\ 0, & t \leq -1. \end{cases} \tag{108}$$

Define a test function φ_ζ on $\mathbb{R}^n$ by

$$\varphi_\zeta(x) = \psi\left(|\zeta|(a-|x|)\right), \quad x \in \mathbb{R}^n. \tag{109}$$

Here $|\zeta| = \sqrt{|\zeta_1|^2 + \cdots + |\zeta_n|^2}$. Note that $\varphi_\zeta = 0$ if

$$\begin{aligned} |\zeta|(a-|x|) &\leq -1 \\ a-|x| &\leq -\frac{1}{|\zeta|} \\ |x| &\geq a + \frac{1}{|\zeta|}, \end{aligned}$$

and $\varphi_\zeta = 1$ if

$$\begin{aligned} |\zeta|(a-|x|) &\geq -\frac{1}{2} \\ a-|x| &\geq -\frac{1}{2|\zeta|} \\ |x| &\leq a + \frac{1}{2|\zeta|}. \end{aligned}$$

So supp $\varphi_\zeta \subset \{|x| \leq a + 1/2|\zeta|\}$. This allows us to write Equation (105) as

$$\hat{u}(\zeta) = \left\langle u(x), \varphi_\zeta(x) e^{-ix\cdot\zeta} \right\rangle. \tag{110}$$

The Fourier-Laplace transform $\hat{u}$ is bounded for $|\zeta| \le 1$. This is because

$$|\hat{u}| = \left| \int_{\text{supp }\varphi_\zeta} u(x)\varphi_\zeta(x) e^{-ix\cdot\zeta}\ \mathrm{d}x \right| \le \sup|\varphi_\zeta| \int |u(x)|\ \mathrm{d}x \le ||u||_{L^1}. \tag{111}$$

On the other hand, for $|\zeta| \ge 1$, we see that supp $\varphi_\zeta \subset \{|x| \le a+1\}$. Therefore, we obtain a semi-norm estimate for $\hat{u}$

$$|\hat{u}(\zeta)| \le C' \sum_{|\alpha|\le N} \sup \left| D_x^\alpha (\varphi_\zeta(x) e^{-ix\cdot\zeta}) \right| \tag{112}$$

for some nonnegative constants C' and N. Expanding the term $D_x^\alpha(\varphi_\zeta(x)e^{-ix\cdot\zeta})$ using Leibniz's Theorem yields the following inequalities:

$$|D_x^\beta \varphi_\zeta(x)| = |D_x^\beta \psi\left(|\zeta|(a-|x|)\right)| \le C_\beta |\zeta|^{|\beta|} \tag{113}$$

for some constants C_β; and

$$|D_x^\gamma e^{-ix\cdot\zeta}| \le |\zeta^\gamma| e^{x\cdot \text{Im }\zeta} \le |\zeta|^{|\gamma|} \exp\left[|\text{Im }\zeta| \left(a + \frac{1}{|\zeta|}\right)\right], \quad x \in \text{supp } \varphi_\zeta. \tag{114}$$

We note that

$$|e^{-ix\cdot\zeta}| = |e^{-ix\cdot(\text{Re }\zeta + i\ \text{Im }\zeta)}| = |e^{-ix\cdot\text{Re }\zeta} e^{x\cdot\text{Im }\zeta}| = |e^{x\cdot\text{Im }\zeta}|$$

and because x is contained in the support of φ_ζ, we have the estimate

$$x \le a + \frac{1}{|\zeta|};$$

hence Inequality (114). All these observations, coupled with a complex number property

$$\frac{|\text{Im }\zeta|}{|\zeta|} \le 1,$$

imply Equation (106), that is, for some nonnegative constants C and N,

$$|\hat{u}(\zeta)| \le C\left(1+|\zeta|\right)^N e^{a|\text{Im }\zeta|}, \quad \zeta \in \mathbb{C}^n.$$

To establish the last assertion, we shall resort to integration by parts. By Equation (59),

$$\zeta^\alpha \hat{u}(\zeta) = \int u(x) D^\alpha(e^{-ix\cdot\zeta})\ \mathrm{d}x = (-1)^{|\alpha|} \int e^{-x\cdot\zeta} D^\alpha u(x)\ \mathrm{d}x, \quad \zeta \in \mathbb{C}^n. \tag{115}$$

Now let $K \subset \mathbb{R}^n$ be a compact subset containing supp $u \subset \{|x| \leq a\}$. Then

$$|\zeta^\alpha \hat{u}(\zeta)| \leq \left| \int_K e^{-ix\cdot\zeta} D^\alpha u(x) \, \mathrm{d}x \right| \leq V_a \sup_{x\in K} |D^\alpha u| \sup_{x\in \mathrm{supp}\ u} |e^{-ix\cdot\zeta}|, \tag{116}$$

where V_a is the measure of the ball $\{|x| \leq a\}$. Absorbing the constants V_a and $\sup |D^\alpha u|$ and writing it as C_K, we get

$$|\zeta^\alpha \hat{u}(\zeta)| \leq C_K e^{a|\mathrm{Im}\ \zeta|}, \tag{117}$$

whence

$$|\hat{u}(\zeta)| \leq \frac{C_K}{|\zeta|^{|\alpha|}} e^{a|\mathrm{Im}\ \zeta|} \tag{118}$$

for all multi-indices α. This illustrates the polynomial decay property of the Fourier-Laplace transform $\hat{u}$, which is Equation (107), with $m = |\alpha|$:

$$|\hat{u}(\zeta)| \leq C_m \left(1 + |\zeta|\right)^{-m} e^{a|\mathrm{Im}\ \zeta|}, \quad \zeta \in \mathbb{C}^n, \ m = 0, 1, \ldots \qquad \square$$

The Malgrange-Ehrenpreis Theorem

Consider the linear differential operator with constant coefficients $P(D)$ defined on $\mathbb{R}^n$. If E is a fundamental solution of P, then $P(D)E = \delta$, whence we have, via Fourier transform and Equation (96),

$$P(\xi)\hat{E} = \widehat{P(D)E} = \hat{\delta} = 1. \tag{119}$$

This reduces the problem to finding a tempered distribution E such that

$$\hat{E}(\xi) = \frac{1}{P(\xi)}. \tag{120}$$

Formally, we are able to recover the distribution E by the Fourier Inversion Theorem:

$$E(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, \frac{\mathrm{d}\xi}{P(\xi)}. \tag{121}$$

However, this division problem poses an obstacle when $P(\xi) = 0$. In the proof of the following theorem, we shall circumvent this problem by regarding $\xi \in \mathbb{R}^n$ as $(\xi', \zeta_n) \in \mathbb{R}^{n-1} \times \mathbb{C}$ and constructing an integral expression for E over a particular integration domain called the "Hörmander's staircase."

Theorem 32 (Malgrange-Ehrenpreis Theorem)**.** *A linear differential operator with constant coefficients has a fundamental solution.*

In this thesis, we shall focus on a special case of the Malgrange-Ehrenpreis Theorem:

Theorem 33. *Let*

$$P(D) = cD_n^m + P_1(D')D_n^{m-1} + \ldots + P_m(D') \tag{122}$$

be a linear differential operator with constant coefficients such that c is nonzero and $D' = (D_1, \ldots, D_{n-1})$ is the $n-1$ tuple of differential operators. Then there exists $E \in \mathcal{D}'(\mathbb{R}^n)$ such that $P(D)E = \delta$.

Proof. The symbol of P can be written as

$$P(\xi) = c\xi_n^m + P_1(\xi')\xi_n^{m-1} + \ldots + P_m(\xi'), \tag{123}$$

where P_j are polynomials. Fix $\bar{\xi}' \in \mathbb{R}^{n-1}$. Then $\zeta_n \mapsto P(\bar{\xi}', \zeta_n)$, with $\zeta_n \in \mathbb{C}$, is a complex polynomial in ζ_n with m zeros by virtue of the Fundamental Theorem of Algebra. Denote the zeros $\lambda_1, \ldots, \lambda_m$. We assert that there exists a real number τ, depending on $\bar{\xi}'$, such that

$$|\tau(\bar{\xi}')| \leq m + 1 \quad \text{and} \quad |\tau(\bar{\xi}') - \text{Im } \lambda_j| > 1, \quad j = 1, 2, \ldots, m. \tag{124}$$

To see this, we shall consider the case when $m = 3$ and resort to proof by contradiction. Under this assumption, the polynomial $\zeta_n \mapsto P(\bar{\xi}', \zeta_n)$ has 3 roots, which shall be denoted λ_1, λ_2, and λ_3 henceforth. Suppose that, for all $\tau(\bar{\xi}')$ such that $|\tau(\bar{\xi}')| \leq 4$,

$$|\tau(\bar{\xi}') - \text{Im } \lambda_j| \leq 1, \quad j = 1, 2, 3.$$

This implies that the length of $\tau(\bar{\xi}')$ containing each Im λ_j is at most 2. Assuming that these intervals are disjoint, we obtain $3 \times 2 = 6$ as the upper bound for the total length of all such $\tau(\bar{\xi}')$. This does not exhaust the entire length of $\tau(\bar{\xi}')$ as in the hypothesis, that is 8.

Consider a neighborhood $U = U(\bar{\xi}')$ of $\bar{\xi}'$ in $\mathbb{R}^{n-1}$. By Rouché's Theorem, if $\xi' \in U(\bar{\xi}')$, then Equation (124) also holds. This is because if $\xi' \in U(\bar{\xi}')$, then the roots λ_j of $P(\xi', \zeta_n)$ are within ϵ_j-neighborhoods of the roots $\overline{\lambda}_j$ of $P(\bar{\xi}', \zeta_n)$. We may take $U(\bar{\xi}')$ to be a cube:

$$U(\bar{\xi}') = \{\xi' : |\xi_j - \bar{\xi}_j| < \delta(\bar{\xi}) \text{ with } \delta < 1 \text{ for all } \bar{\xi} \in \mathbb{R}^{n-1},\ j = 1, 2, \ldots, n\}. \tag{125}$$

These cubes form an open covering of $\mathbb{R}^{n-1}$. Now consider closed balls $B_j = \{\xi' \in \mathbb{R}^{n-1} : |\xi'| \leq j\}$. Clearly $\mathbb{R}^{n-1} = \cup_{j=1}^{\infty} B_j$. Since each B_j is compact, by Heine-Borel Theorem, every open cover of B_j has a finite subcover. Let A_j be the subcover of B_j. Each A_j encompasses finitely many cubes covering B_j. Hence one may extract a countable subcover of $\mathbb{R}^{n-1}$ via

$$\bigcup_{j=1}^{\infty} A_j \tag{126}$$

and order as $U_j = U(\bar{\xi}'_{(j)})$ with $j = 1, 2, \ldots$. Then, we define mutually disjoint open sets Ω_j by

$$\Omega_1 = U_1, \quad \Omega_2 = U_2 \setminus \overline{\Omega}_1, \quad \ldots, \quad \Omega_k = U_k \setminus \bigcup_{j=1}^{k-1} \overline{\Omega}_j, \quad \ldots. \tag{127}$$

Note that the union of their closures is $\mathbb{R}^{n-1}$ and any compact set meets only a finite number of them.

Now, we shall consider the following sets, for $j = 1, 2, \ldots$, that constitute the "Hörmander's staircase:"

$$\Gamma_j = \{\zeta \in \mathbb{C}^n : \zeta' = \xi' \in \Omega_j,\ \zeta_n = \xi_n + i\tau(\bar{\xi}'_{(j)}),\ \xi_n \in \mathbb{R}\}. \tag{128}$$

To visualize these sets, consider the case of $n = 2$. Then $\Omega_j \subset \mathbb{R}$ and $\zeta = (\xi', \zeta_2)$. Since ξ_n runs from $-\infty$ to ∞ and τ is a positive constant, we see that Γ_j is a strip in the 3-space of ξ', real part of ζ_2, and imaginary part of ζ_2. These sets Γ_j avoid the zeros of the polynomial P.

Note that Equation (124) holds in each U_j, whence

$$|P(\zeta)| = |c||\zeta - \lambda_1| \cdots |\zeta - \lambda_m| \geq |c||\text{Im } \zeta - \text{Im } \lambda_1| \cdots |\text{Im } \zeta - \text{Im } \lambda_m| \geq |c|, \quad \zeta \in \Gamma_j. \tag{129}$$

We shall now define a linear form $E : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ via the formula

$$\langle E, \check{\varphi} \rangle = \sum_{j=1}^{\infty} \int_{\Gamma_j} \frac{\hat{\varphi}(\zeta)}{P(\zeta)}\, \mathrm{d}\zeta, \quad \varphi \in C_c^\infty(\mathbb{R}^n). \tag{130}$$

The pairing of Equation (121) with a test function $\check{\varphi}$ motivates the definition above:

$$\langle E, \check{\varphi} \rangle = \int \sum_{j=1}^{\infty} \int_{\Gamma_j} e^{ix\cdot\zeta} \frac{\check{\varphi}(\zeta)}{P(\zeta)}\, \mathrm{d}\zeta\ \mathrm{d}x = \sum_{j=1}^{\infty} \int_{\Gamma_j} \left[\int e^{ix\cdot\zeta} \varphi(-\zeta)\ \mathrm{d}x \right] \frac{\mathrm{d}\zeta}{P(\zeta)} = \sum_{j=1}^{\infty} \int_{\Gamma_j} \frac{\hat{\varphi}(\zeta)}{P(\zeta)}\, \mathrm{d}\zeta.$$

The integral in Equation (130) is well defined because $|P(\zeta)| \geq |c|$ is bounded away from zero, as described in Equation (129). The sum converges as each integrand has an upper bound by the estimate in the Paley-Wiener-Schwartz Lemma:

$$|\hat{\varphi}(\xi', \xi_n + i\tau)| \leq C_\varphi \left(1 + |\xi'| + |\xi_n + i\tau|\right)^{-(n+1)} e^{a|\tau|}. \tag{131}$$

Next, we want to show that E is a member of $\mathcal{D}'(\mathbb{R}^n)$. The constant C in Equation (131) is

$$C_\varphi = V_a \sup_{x\in K} |D^\alpha \varphi|, \quad |\alpha| = n+1 \tag{132}$$

as suggested by Equation (116), where V_a is the measure of the ball $\{|x| \le a\}$ and $K \subset \mathbb{R}^n$ is a compact subset containing supp φ. Then,

$$\begin{aligned}
|\langle E, \check{\varphi}\rangle| = \left|\sum_{j=1}^{\infty} \int_{\Gamma_j} \frac{\hat{\varphi}(\zeta)}{P(\zeta)}\, \mathrm{d}\zeta\right| &\le \frac{1}{c}\sum_{j=1}^{\infty} \int_{\Gamma_j} |\hat{\varphi}|\, \mathrm{d}\zeta \\
&\le \frac{C_\varphi}{c} e^{a|\tau|} \sum_{j=1}^{\infty} \int_{\xi'\in\Omega_j, \xi_n \in \mathbb{R}} (1+|\xi'|+|\xi_n+i\tau|)^{-(n+1)}\ \mathrm{d}\xi'\ \mathrm{d}\zeta_n \\
&\le \frac{C_\varphi}{c} e^{a|\tau|} \sum_{j=1}^{\infty} \int_{\xi'\in\Omega_j, \xi_n \in \mathbb{R}} (1+|\xi'|+|\xi_n|)^{-(n+1)}\ \mathrm{d}\xi'\ \mathrm{d}\xi_n \\
&= \frac{C_\varphi}{c} e^{a|\tau|} \int_{\mathbb{R}^n} (1+|\xi'|+|\xi_n|)^{-(n+1)}\ \mathrm{d}x.
\end{aligned}$$

The last equality is due to the fact that Ω_j are mutually disjoint, as indicated in Equation (127). Finally, the integral

$$\int_{\mathbb{R}^n} (1+|\xi'|+|\xi_n|)^{-(n+1)}\ \mathrm{d}x$$

converges to a finite number m. Hence

$$|\langle E, \check{\varphi}\rangle| \le C_\varphi e^{a|\tau|}\left(\frac{m}{c}\right). \tag{133}$$

But the constant C_φ in Equation (132) contains the semi-norm of φ. Thus the following semi-norm estimate

$$|\langle E, \check{\varphi}\rangle| \le C' \sup_{x\in K} |D^\alpha \varphi| \tag{134}$$

holds, whereby the constant C' is

$$C' = e^{a|\tau|}\left(\frac{mV_a}{c}\right)$$

Therefore $E \in \mathcal{D}'(\mathbb{R}^n)$.

Finally, we need to establish that E is a fundamental solution, that is $PE = \delta$. We note that if $\varphi \in C_c^\infty(\mathbb{R}^n)$, then

$$\langle PE, \check{\varphi}\rangle = \langle E, P(-D)\check{\varphi}\rangle = \langle E, (P\varphi)\check{}\,\rangle.$$

From Equation (130), we see that

$$\langle PE, \check{\varphi}\rangle = \int_\Gamma \frac{\widehat{P\varphi}(\zeta)}{P(\zeta)}\ \mathrm{d}\zeta = \int_\Gamma P(\zeta)\frac{\hat{\varphi}(\zeta)}{P(\zeta)}\ \mathrm{d}\zeta = \int_\Gamma \hat{\varphi}(\zeta)\ \mathrm{d}\zeta.$$

For each j, the integral along each step of the Hörmander's staircase can be written fully as

$$\int_{\Gamma_j} \hat{\varphi}(\zeta)\ \mathrm{d}\zeta = \int_{\xi' \in \Omega_j, \xi_n \in \mathbb{R}} \hat{\varphi}\left(\xi', \xi_n + i\tau\right)\ \mathrm{d}\xi'\ \mathrm{d}\xi_n. \tag{135}$$

Now fix $\bar{\xi}' \in \mathbb{R}^{n-1}$. Then the integrand $\hat{\varphi}$ in Equation (135) is integrated over the complex plane of ζ_n. Moreover, τ is a constant — so the contour of integration is a horizontal line on the ζ_n-plane positioned at $\mathrm{Im}\ \zeta_n = \tau$, where $\xi_n = \mathrm{Re}\ \zeta_n$ runs from $-\infty$ to ∞.

Let γ denote this contour of integration. Then the integral in Equation (135), with fixed $\xi' = \bar{\xi}'$ is

$$I_1 = \int_{\bar{\xi}' \in \Omega_j, \xi_n \in \mathbb{R}} \hat{\varphi}\left(\bar{\xi}', \xi_n + i\tau\right)\ \mathrm{d}\xi_n = \int_{\gamma} \hat{\varphi}(\bar{\xi}', \zeta_n)\ \mathrm{d}\zeta_n \tag{136}$$

We shall argue via Cauchy Theorem that $I_1 = I_2$, which is the integral

$$I_2 = \int_{\mathbb{R}} \hat{\varphi}(\bar{\xi}', \xi_n)\ \mathrm{d}\xi_n. \tag{137}$$

To do so, consider the rectangle $\mathcal{R} = [-R, R] \times [0, i\tau]$ on the complex plane of ζ_n. Let the boundary $\partial\mathcal{R}$ of the rectangle $\mathcal{R}$ be represented by four contours. Let γ_1 be the horizontal line at $\mathrm{Im}\ \zeta_n = \tau$ running from $\xi_n = -R$ to $\xi_n = R$ and γ_3 be the horizontal line directly below γ_1, at $\mathrm{Im}\ \zeta_n = 0$, running from $\xi_n = R$ to $\xi_n = -R$. Furthermore, let γ_2 be the vertical line at $\xi_n = R$ running from $\mathrm{Im}\ \zeta_n = \tau$ to $\mathrm{Im}\ \zeta_n = 0$, and γ_4 be the vertical line at $\xi_n = -R$ running from $\mathrm{Im}\ \zeta_n = 0$ to $\mathrm{Im}\ \zeta_n = \tau$.

It follows that $\partial\mathcal{R} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. By construction, $\hat{\varphi}$ is analytic in $\mathcal{R}$. Thus the Cauchy Theorem applies, asserting that

$$\int_{\partial\mathcal{R}} \hat{\varphi}\ \mathrm{d}\zeta_n = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} \hat{\varphi}\ \mathrm{d}\zeta_n = 0. \tag{138}$$

However, as $R \to \infty$, integrals over γ_2 and γ_4 vanish due to the Paley-Wiener-Schwartz estimate in Equation (131), that is $|\hat{\varphi}| \to 0$ as R increases without bound. Hence, from Equation (138), we have

$$\lim_{R\to\infty} \int_{\gamma_1+\gamma_3} \hat{\varphi}\ \mathrm{d}\zeta_n = \lim_{R\to\infty} \int_{-R}^{R} \hat{\varphi}(\bar{\xi}', \xi_n + i\tau)\ \mathrm{d}\xi_n + \lim_{R\to\infty} \int_{R}^{-R} \hat{\varphi}(\bar{\xi}', \xi_n)\ \mathrm{d}\xi_n = 0. \tag{139}$$

Equivalently, we have

$$\lim_{R\to\infty} \int_{-R}^{R} \hat{\varphi}(\bar{\xi}', \xi_n + i\tau)\ \mathrm{d}\xi_n = \lim_{R\to\infty} \int_{-R}^{R} \hat{\varphi}(\bar{\xi}', \xi_n)\ \mathrm{d}\xi_n. \tag{140}$$

We have thus established that $I_1 = I_2$.

Now, we can conclude that

$$\langle PE, \check{\varphi} \rangle = \int_\Gamma \hat{\varphi}(\zeta) \, \mathrm{d}\zeta = \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \, \mathrm{d}\xi = \varphi(0) = \check{\varphi}(0) = \langle \delta, \check{\varphi} \rangle \,, \quad \varphi \in C_c^\infty(\mathbb{R}^n). \tag{141}$$

Replacing $\check{\varphi}$ with φ completes the proof of the theorem:

$$\langle PE, \varphi \rangle = \langle \delta, \varphi \rangle \,, \quad \varphi \in C_c^\infty(\mathbb{R}^n). \tag{142}$$

That is, E is the fundamental solution to P. □

References

[1] F. Gerald Friedlander and Mark Joshi, *Introduction to the theory of distributions*, Cambridge University Press, Cambridge, UK, 1998.